Mathematics Department Stanford University
Math 51H Mid-Term 1

October 15, 2013

Unless otherwise indicated, you can use results
covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

75 minutes

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________

1 (a) (3 points): Give the definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,\ldots}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and use your definition to prove $\ell \geq 0$, assuming that the limit ℓ exists and that $a_n \geq 0\,\forall n$.

(b) (3 points): Suppose that S is a non-empty subset of $\mathbb{R}$ which is bounded above, and let $\alpha = \sup S$.

(i) Prove that for each $\varepsilon > 0$ there is $x \in S$ with $x > \alpha - \varepsilon$.

(ii) Prove that there is a sequence $\{x_n\}_{n=1,2,\ldots}$ with $x_n \in S$ for each n and $\lim x_n = \alpha$.

Name:________________________

2 (a) (3 points): Suppose $\underset{\sim}{a}, \underset{\sim}{b}$ are distinct vectors in $\mathbb{R}^n$.

(i) Give the definition of "the line ℓ through $\underset{\sim}{a}$ parallel to $\underset{\sim}{b} - \underset{\sim}{a}$," and find the vector $\underset{\sim}{v} \in \ell$ which is equi-distant from $\underset{\sim}{a}, \underset{\sim}{b}$ (i.e. $\|\underset{\sim}{v} - \underset{\sim}{a}\| = \|\underset{\sim}{v} - \underset{\sim}{b}\|$).

(ii) If $\underset{\sim}{v}$ is as in (i) and $\|\underset{\sim}{a}\| = \|\underset{\sim}{b}\|$, prove $\underset{\sim}{v} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$.

(b) (3 points): Prove that $2\,|\underset{\sim}{x} \cdot \underset{\sim}{y}|\,\|\underset{\sim}{x}\|^2 \leq \|\underset{\sim}{x}\|^6 + \|\underset{\sim}{y}\|^2$ for all vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$.

3 (a) (4 points): Suppose

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 & 2 \end{pmatrix}$$

Find (i) a basis for the null space $N(A)$ of A (show all row operations!), and (ii) a basis for the column space $C(A)$.

Make sure you justify your results by referring to the appropriate results from lecture.

3 (b) (3 points): Suppose $V \subset \mathbb{R}^n$ is a non-trivial subspace of dimension k. Give the proof that any k vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k \in V$ which span V (i.e. $V = \text{span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k\}$) must automatically be a basis for V.

Name:________________________

4 (a) (3 points): Suppose A is an $m \times n$ matrix and $\underset{\sim}{b} \in \mathbb{R}^m$. (i) Give the proof that $A\underset{\sim}{x} = \underset{\sim}{b}$ has at least one solution $\underset{\sim}{x} \in \mathbb{R}^n \iff \underset{\sim}{b} \in C(A)$, and (ii) In case $m = n$ and $N(A) = \{\underset{\sim}{0}\}$, prove that $A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution for each $\underset{\sim}{b} \in \mathbb{R}^n$.

Hint for (ii): Start by applying the rank/nullity theorem.

4 (b) (3 points): If V is a subspace of $\mathbb{R}^n$, give the definition of $V^\perp$. Prove (i) that $V^\perp$ is a subspace, and (ii) that $V \cap V^\perp = \{\underset{\sim}{0}\}$.

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