Mathematics Department Stanford University
Math 51H – The Fundamental Theorem of Algebra

As a preliminary we need the following maximum principle for harmonic functions, which is of great importance in itself; we only state and prove the maximum principle on a ball, but the reader should note that an analogous theorem (with essentially identical proof) holds on any bounded open subset of $\mathbb{R}^n$.

In the statement, and subsequently, we use the notation

$$B_R = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| < R\}, \quad \overline{B}_R = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| \le R\}, \quad \partial B_R = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| = R\}.$$

Theorem (Maximum Principle for Harmonic Functions.) *Suppose $u : \overline{B}_R \to \mathbb{R}$ is continuous and $u|B_R$ is C^2 and harmonic (i.e. $\Delta u = \sum_{j=1}^n D_j D_j u \equiv 0$) on B_R. Then $\max_{\overline{B}_R} u = \max_{\partial B_R} u$.*

Proof: Let $\varepsilon > 0$ be arbitrary and define

$$v(\underline{x}) = u(\underline{x}) + \varepsilon\|\underline{x}\|^2, \quad \underline{x} \in \overline{B}_R,$$

and observe that then $u \le v$ in $\overline{B}_R$ and $v \equiv u + \varepsilon R^2$ on ∂B_R.

We claim $\max_{\overline{B}_R} v = \max_{\partial B_R} v (= \max_{\partial B_R} u + \varepsilon R^2)$. If this is false then there is a point $\underline{a} \in B_R$ with $v(\underline{a}) = \max_{\overline{B}_R} v > M + \varepsilon R^2$, where, here and subsequently, we take $M = \max_{\partial B_R} u$. Pick $\delta > 0$ with $B_\delta(\underline{a}) \subset B_R$, Then, for each $j = 1, \ldots, n$, $v(\underline{a} + t\underline{e}_j)$ is a C^2 function of t for $|t| < \delta$ and takes its maximum at $t = 0$. Hence from 1-variable calculus we have $\frac{d}{dt}[v(\underline{a} + t\underline{e}_j)]|_{t=0} = 0$ and $\frac{d^2}{dt^2}[v(\underline{a} + t\underline{e}_j)]|_{t=0} \le 0$. But of course by definition of partial derivatives, this says exactly that $D_j v(\underline{a}) = 0$ and $D_j D_j v(\underline{a}) \le 0$. By taking the sum over j the latter inequality implies $\Delta v(\underline{a}) \le 0$, whereas by direct computation we have $\Delta v = \Delta u + 2n\varepsilon \equiv 2n\varepsilon > 0$ at each point of B_R, a contradiction. So indeed $\max_{\overline{B}_R} v = M + \varepsilon R^2$ and hence $u(\underline{x}) \le v(\underline{x}) \le M + \varepsilon R^2$ for each $\underline{x} \in \overline{B}_R$. Since $\varepsilon > 0$ is arbitrary this shows that $u(\underline{x}) \le M$ for each $\underline{x} \in \overline{B}_R$.

Theorem (Fundamental Theorem of Algebra.) *Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of the complex variable $z = x + iy$ (where $a_0, \ldots, a_{n-1}$ are given complex numbers). Then there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $P(z) \equiv (z - \lambda_1)\cdots(z - \lambda_n)$ for all $z \in \mathbb{C}$. (Thus $\lambda_1, \ldots, \lambda_n$ are the roots of $P(z) = 0$.)*

Remarks: Before we start the proof we make 3 remarks about such polynomials $P(z)$:

(i) $\lambda \in \mathbb{C} \Rightarrow P(z) - P(\lambda) = z^n - \lambda^n + \sum_{j=1}^{n-1} a_j(z^j - \lambda^j)$ and hence using the formula $z^j - \lambda^j = (z - \lambda)(z^{j-1} + \lambda z^{j-2} + \cdots + \lambda^{j-2}z + \lambda^{j-1})$ for $j \ge 2$, we obtain $P(z) - P(\lambda) = (z - \lambda)Q(z)$, where $Q(z) = z^{n-1} + \lambda z^{n-2} + \cdots + \lambda^{n-1} + \sum_{j=1}^{n-1} a_j(z^{j-1} + \lambda z^{j-2} + \cdots + \lambda^{j-2}z + \lambda^{j-1})$ is a polynomial of degree $n - 1$. In particular if $P(\lambda_1) = 0$ (i.e. if λ_1 is a root of the equation $P(z) = 0$) then the above with $\lambda = \lambda_1$ implies $P(z) \equiv (z - \lambda_1)Q(z)$ for all $z \in \mathbb{C}$. Thus to prove the above fundamental theorem of algebra it suffices to prove we can always find *one root*, because then we can use induction on n with the inductive hypothesis that the theorem is true with $n - 1$ in place of n for any $n \ge 2$ (which guarantees that $Q(z) = (z - \lambda_2)\cdots(z - \lambda_n)$ and hence $P(z) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$ as required).

(ii) As we proved in Q.8 of hw9, the real and imaginary parts u, v of P satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$, and in particular are harmonic functions.

(iii) At points where $P(z) \neq 0$ we claim that the real and imaginary parts of $\frac{1}{P(z)}$ satisfy the Cauchy-Riemann equations, and hence are also harmonic. Check:

$$\frac{1}{P(z)} = \frac{1}{u+iv} = S + iT \text{ with } S, T \text{ real } \Rightarrow S = \frac{u}{u^2+v^2},\ T = \frac{-v}{u^2+v^2},$$

so by using the quotient rule for taking derivatives we have

$$S_x = \frac{(u^2+v^2)u_x - 2u(uu_x + vv_x)}{(u^2+v^2)^2} = \frac{u_x(v^2-u^2) - 2uvv_x}{(u^2+v^2)^2}$$

$$T_y = \frac{-(u^2+v^2)v_y + 2v(uu_y + vv_y)}{(u^2+v^2)^2} = \frac{v_y(v^2-u^2) + 2uvu_y}{(u^2+v^2)^2}$$

and since $u_x = v_y$ and $u_y = -v_x$ (by Remark (ii) above), we thus have $S_x = T_y$. Similarly (exercise) $S_y = -T_x$.

Proof of the Fundamental Theorem of Algebra: As pointed out in Remark (i) above, it suffices to prove that there is one root of $P(z) = 0$; that is, we simply have to show that $P(z)$ vanishes somewhere. Suppose on the contrary that $P(z) \neq 0$ for each $z \in \mathbb{C}$. Then by Remark (iii) above the real and imaginary parts S, T of $1/P(z)$ are harmonic on $\mathbb{R}^2$. Also by the triangle inequality

$$|P(z)| = |z^n + \textstyle\sum_{j=0}^{n-1} a_j z^j| \geq |z^n| - |\textstyle\sum_{j=0}^{n-1} a_j z^j| \geq |z^n| - \textstyle\sum_{j=0}^{n-1} |a_j z^j| = |z|^n - \textstyle\sum_{j=0}^{n-1} |a_j|\,|z|^j,$$

and so

$$|z| = R \geq 1 \Rightarrow |P(z)| \geq R^n - \textstyle\sum_{j=0}^{n-1} |a_j| R^j \geq R^n(1 - R^{-1}\textstyle\sum_{j=0}^{n-1} |a_j|),$$

and hence

$$|z| = R \geq \max\{1, 2\textstyle\sum_{j=0}^{n-1} |a_j|\} \Rightarrow |P(z)| \geq \frac{R^n}{2} \Rightarrow \frac{1}{|P(z)|} \leq \frac{2}{R^n},$$

hence

$$\varepsilon > 0 \text{ and } |z| = R \geq \max\{1, 2\textstyle\sum_{j=0}^{n-1} |a_j|, \left(\frac{2}{\varepsilon}\right)^{1/n}\} \Rightarrow \frac{1}{|P(z)|} \leq 2R^{-n} \leq \varepsilon.$$

So take any $\varepsilon > 0$ and let $R_0 = \max\{1, 2\sum_{j=0}^{n-1} |a_j|, (\frac{2}{\varepsilon})^{1/n}\}$. Since $\max\{|S|, |T|\} \leq \sqrt{S^2+T^2} = \frac{1}{|P(z)|}$ the above inequality tells us that

$$|z| = R \geq R_0 \Rightarrow |S|,\ |T| \leq \varepsilon.$$

Since S, T are harmonic we can then use the maximum principle to conclude

$$\max_{\overline{B}_R} S = \max_{\partial B_R} S \leq \varepsilon, \quad R \geq R_0,$$

and (applying the maximum principle to $-S$)

$$\max_{\overline{B}_R} -S = \max_{\partial B_R} -S \leq \varepsilon, \quad R \geq R_0.$$

Thus $|S| \leq \varepsilon$ on $\overline{B}_R \,\forall\, R \geq R_0$ and hence $\sup_{\mathbb{R}^2} |S| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary this implies $S \equiv 0$. Similarly $T \equiv 0$, so both S and T vanish identically, which of course is impossible because $1/P(z) = S + iT$ is in fact *never* zero. Thus $P(z)$ must have a zero somewhere in $\mathbb{C}$ and the proof is complete.