Mathematics Department Stanford University
Math 51H – Contraction mapping theorem and ODEs

The contraction mapping theorem concerns maps $f : X \to X$, (X, d) a metric space, and their fixed points. A point x is a **fixed point** of f if $f(x) = x$, i.e. f fixes x. A **contraction mapping** is a map $f : X \to X$ such that there is $\theta \in (0, 1)$ such that $d(f(x), f(y)) \leq \theta d(x, y)$ for all $x, y \in X$.

Theorem 1 *Suppose X is a complete metric space, and $f : X \to X$ is a contraction mapping. Then f has a unique fixed point x.*

Remark 1 *Note that if (Z, d) is a complete metric space and X is a closed subset, then X is complete (with the relative metric). Indeed, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X, then it is such in Z, thus it converges to some $x \in Z$. But then x is a limit point of X, and X is closed, so $x \in X$, proving the desired completeness (every Cauchy sequence in X converges to a point in X).*

Remark 2 *Note that any contraction mapping is (uniformly!) continuous; one can take $\delta = \varepsilon$ in the definition of continuity.*

Proof of the Theorem: First suppose x, x' are fixed points of f. Then $f(x) = x$, $f(x') = x'$, so

$$d(x, x') = d(f(x), f(x')) \leq \theta d(x, x'),$$

so $(1 - \theta)d(x, x') \leq 0$, so $d(x, x') \leq 0$, so $d(x, x') = 0$, so $x = x'$, showing uniqueness.

We now turn to existence. To do so, take an arbitrary $x_0 \in X$, and define x_n inductively: $x_{n+1} = f(x_n)$, $n \geq 0$. We claim that $\{x_n\}_{n=0}^{\infty}$ is Cauchy. To see this, notice first that for all $n \geq 1$,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \theta d(x_n, x_{n-1}),$$

so by induction,

$$d(x_{n+1}, x_n) \leq \theta^n d(x_1, x_0).$$

Thus, for $n > m$, using the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + \ldots + d(x_{m+1}, x_m) \leq (\theta^{n-1} + \theta^{n-2} + \ldots + \theta^m)d(x_1, x_0).$$

Summing the finite geometric series,

$$d(x_n, x_m) \leq \theta^m(1 + \theta + \ldots + \theta^{n-m-1})d(x_1, x_0) = \theta^m d(x_1, x_0)\frac{1 - \theta^{n-m}}{1 - \theta} \leq \theta^m \frac{d(x_1, x_0)}{1 - \theta}.$$

Since $\lim_{m\to\infty} \theta^m = 0$, we have that given $\varepsilon > 0$ there exists N such that $m \geq N$ implies $\theta^m < \varepsilon\frac{1-\theta}{1+d(x_1,x_0)}$, and thus for $n, m \geq N$, $d(x_n, x_m) < \varepsilon$, proving the Cauchy claim.

But X is complete, so $x = \lim_{n\to\infty} x_n$ exists. Since f is continuous and $\lim x_n = x$, sequential continuity shows that $\lim f(x_n) = f(x)$. But $f(x_n) = x_{n+1}$, so $\lim_{n\to\infty} x_{n+1} = f(x)$. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$, we deduce that $f(x) = x$, so x is a fixed point of f as claimed. □

We now use this in the simplest ODE setting. An ODE (ordinary differential equation) is an equation of the form

$$x'(t) = F(t, x(t)),\ x(t_0) = x_0 \tag{1}$$

where $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and one is looking for a solution x at least on an interval near t_0, say $[t_0 - \delta, t_0 + \delta]$, $\delta > 0$, so $x \in C^1([t_0 - \delta, t_0 + \delta])$. We impose the minimal requirement that F be continuous for the following discussion.

Rather than considering the ODE directly, we rewrite it as an integral equation using the fundamental theorem of calculus:

$$x(t) = x(t_0) + \int_{t_0}^{t} x' = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau.$$

Note that any solution of the original ODE solves the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau, \tag{2}$$

and conversely if $x \in C^0([t_0 - \delta, t_0 + \delta])$ merely solving (2), then in fact x is C^1 by the fundamental theorem of calculus (since F is continuous, being the composite of continuous functions!), and solves the ODE.

Thus, from now on we consider (2). We consider the equation as a fixed point claim for a map $T : X \to X$, $X = C^0([t_0 - \delta, t_0 + \delta])$. Concretely, let

$$Tx(t) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau;$$

so certainly $T : X \to X$. Now, as $d(x, y) = \sup\{\|x(t) - y(t)\| :\ t \in [t_0 - \delta, t_0 + \delta]\}$, we have

$$\begin{aligned} d(Tx, Ty) &= \sup\Big\{\Big\|(x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau) - (x_0 + \int_{t_0}^{t} F(\tau, y(\tau))\, d\tau)\Big\| :\ t \in [t_0 - \delta, t_0 + \delta]\Big\} \\ &= \sup\Big\{\Big\| \int_{t_0}^{t} (F(\tau, x(\tau)) - F(\tau, y(\tau)))\, d\tau)\Big\| :\ t \in [t_0 - \delta, t_0 + \delta]\Big\}. \end{aligned}$$

So let us suppose that F is **globally Lipschitz** in the second variable, i.e. for some $\delta_0 > 0$,

$$\|F(t, x) - F(t, y)\| \le M\|x - y\|,\ x, y \in \mathbb{R}^n, t \in [t_0 - \delta_0, t_0 + \delta_0]. \tag{3}$$

Then, if $\delta \le \delta_0$, $t \ge t_0$ (with a similar calculation if $t < t_0$)

$$\begin{aligned} \Big\|\int_{t_0}^{t} (F(\tau, x(\tau)) - F(\tau, y(\tau)))\, d\tau)\Big\| &\le \int_{t_0}^{t} \|(F(\tau, x(\tau)) - F(\tau, y(\tau))\|\, d\tau \le \int_{t_0}^{t} M\|x(\tau) - y(\tau)\|\, d\tau \\ &\le \int_{t_0}^{t} M \sup\{\|x(s) - y(s)\| :\ s \in [t_0 - \delta, t_0 + \delta]\}\, d\tau \\ &= M(t - t_0) \sup\{\|x(s) - y(s)\| :\ s \in [t_0 - \delta, t_0 + \delta]\} \\ &\le M\delta \sup\{\|x(s) - y(s)\| :\ s \in [t_0 - \delta, t_0 + \delta]\} = M\delta d(x, y). \end{aligned} \tag{4}$$

Thus,

$$d(Tx, Ty) \le M\delta d(x, y),$$

so it is a contraction mapping provided $\delta < \frac{1}{M}$.

So take $\delta < 1/M$, $\delta \le \delta_0$. Then T is a contraction mapping on the complete metric space $C^0([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n)$ (continuous maps from $[t_0 - \delta, t_0 + \delta]$ to $\mathbb{R}^n$), with completeness shown in Problem set 9, Problem 7. Thus, it has a unique fixed point $x \in C^0([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n)$, which as explained means that the ODE has a unique solution on $[t_0 - \delta, t_0 + \delta]$:

Theorem 2 *Suppose F is globally Lipschitz on $[t_0 - \delta_0, t_0 + \delta_0] \times \mathbb{R}^n$ in the sense of* (3)*. Then there is $\delta > 0$, $\delta \le \delta_0$ such that the ODE* (1) *has a unique C^1 solution on $[t_0 - \delta, t_0 + \delta]$.*

This is the simplest local existence and uniqueness theorem for ODE. The only unsatisfactory assumption in it is (3). Notice that the estimate of (3) is a very reasonable assumption *locally*, namely if it is replaced by

$$\|F(t, x) - F(t, y)\| \le M\|x - y\|,\ x, y \in \overline{B}_R(0), t \in [t_0 - \delta_0, t_0 + \delta_0], \tag{5}$$

where $\overline{B}_R(0)$ is the closed ball of radius R in $\mathbb{R}^n$: $\overline{B}_R(0) = \{x \in \mathbb{R}^n : \|x\| \leq R\}$. Indeed, in this case it follows from F being C^1, since by the fundamental theorem of calculus, just as in our proof of Taylor's theorem,

$$F(t,x) - F(t,y) = \sum_{j=1}^{n} (x_j - y_j) \int_0^1 D_j F(t, y + s(x-y))\, ds = \int_0^1 DF(t, y + s(x-y))(x-y)\, ds,$$

where D_j, D denote derivatives in the second slot. Thus,

$$\begin{aligned}\|F(t,x) - F(t,y)\| &\leq \int_0^1 \Big\| DF(t, y + s(x-y))(x-y) \Big\|\, ds \\ &\leq \int_0^1 \Big\| DF(t, y + s(x-y)) \Big\| \|x-y\|\, ds = \|x-y\| \int_0^1 \Big\| DF(t, y + s(x-y)) \Big\|\, ds,\end{aligned}$$

so if F is C^1, so $D_j F$ are bounded on the compact set $[t_0 - \delta_0, t_0 + \delta_0] \times \overline{B}_R(0)$,

$$\|F(t,x) - F(t,y)\| \leq C\|x-y\|,\ C = \sup\{\|DF(z)\| :\ z \in [t_0 - \delta_0, t_0 + \delta_0] \times \overline{B}_R(0)\}.$$

The more natural ODE theorem is then:

Theorem 3 *Suppose F is locally Lipschitz on $[t_0 - \delta_0, t_0 + \delta_0] \times \overline{B}_R(0)$, $R > 0$, in the sense of* (5). *Then there is $\delta > 0$, $\delta \leq \delta_0$ such that for $x_0 \in \overline{B}_{R/2}(0)$ the ODE* (1) *has a unique C^1 solution on $[t_0 - \delta, t_0 + \delta]$ with $\|x(t)\| \leq R$ on $[t_0 - \delta, t_0 + \delta]$.*

Proof: Let $C_0 = \sup\{\|F(z)\| : \in [t_0 - \delta_0, t_0 + \delta_0] \times \overline{B}_R(0)\}$. Then for $\|x_0\| \leq R/2$, $\delta \leq \delta_0$, $x \in C^0([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n)$ with $\|x(t)\| \leq R$ for $t \in [t_0 - \delta, t_0 + \delta]$, and for $t \geq t_0$ (with a similar formula for $t < t_0$),

$$\|(Tx)(t)\| \leq \|x_0\| + \int_{t_0}^{t} \|F(\tau, x(\tau)\|\, d\tau \leq R/2 + C_0(t - t_0) \leq R/2 + C_0\delta.$$

Thus, if $\delta < R/(2C_0 + 1)$, then

$$\|(Tx)(t)\| \leq R,\ t \in [t_0 - \delta, t_0 + \delta].$$

Thus, if we let $X = \{x \in C^0([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n) :\ \sup \|x\| \leq R\}$, then $T : X \to X$ provided $\delta < R/(2C_0 + 1)$ (and $\delta \leq \delta_0$).

Further, for $x \in X$, we have by (5)

$$\|F(t, x(t)) - F(t, y(t))\| \leq M\|x(t) - y(t)\|,\ t \in [t_0 - \delta, t_0 + \delta],$$

so the calculation of (4) applies. This gives that if in addition $\delta < M^{-1}$ then T is a contraction mapping, and thus the contraction mapping theorem gives a unique fixed point. This gives a unique solution to the ODE with the property that $\|x(t)\| \leq R$ for $t \in [t_0 - \delta, t_0 + \delta]$, and proves the theorem.
□